Session 4 : solutions

Exercise 1

$$Z = \sum_{S_1=\pm1,0} \cdots \sum_{S_N=\pm1,0} e^{\beta J \sum_{i=1}^{N} S_i S_{i+1} - \beta D \sum_i S_i^2} \underset{\text{periodic b.c.}}{=}$$

$$= \sum_{S_1=\pm1,0} \cdots \sum_{S_N=\pm1,0} T(S_1,S_2) \cdots T(S_{N-1},S_N)\, T(S_N,S_1) =$$

$$= \mathrm{Tr}\, \underline{\underline{T}}^N = \underset{\substack{\text{largest}\\ \text{eigenvalue}}}{\lambda^N_{MAX}} + \underset{\text{intermediate}}{\lambda^N_{int}} + \underset{\substack{\text{minimal}\\ \text{eigenvalue}}}{\lambda^N_{min}}$$

There are three eigenvalues because, after writing

$$H = -J \sum_{i=1}^{N} S_i S_{i+1} + \frac{D}{2} \sum_i \left(S_i^2 + S_{i+1}^2 \right)$$

we have

$$\underline{\underline{T}} = \begin{array}{c c} & \begin{array}{ccc} + & 0 & - \end{array} \\ \begin{array}{c} + \\ 0 \\ - \end{array} & \begin{pmatrix} e^{\beta(J-D)} & e^{-\beta\frac{D}{2}} & e^{-\beta(J-D)} \\ e^{-\beta\frac{D}{2}} & 1 & e^{-\beta\frac{D}{2}} \\ e^{-\beta(J-D)} & e^{-\beta\frac{D}{2}} & e^{\beta(J-D)} \end{pmatrix} \end{array}$$

it is a 3×3 matrix

$\Rightarrow$ three eigenvalues

No need to compute further analytically.

Exercise 2:

Use $s_i s_j = (s_i - m)(s_j - m) + m(s_i + s_j) + m^2$

so that

$$H_{MF} = -(Jzm + h)\sum_i s_i + D \sum_i s_i^2$$

Then

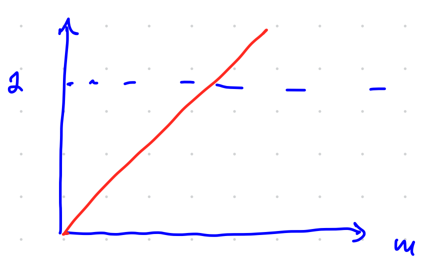
$$Z_{MF} = \sum_{\{s_i\}} \prod_i e^{\beta(Jzm+h)s_i - \beta D s_i^2} =$$

$$= \left[1 + 2e^{-\beta D} \cosh(\beta(Jzm+h)) \right]^N$$

and

$$m = \frac{2e^{-\beta D} \sinh(\beta(Jzm+h))}{1 + 2e^{-\beta D}\cosh(\beta(Jzm+h))}$$

Set $h = 0$



Let's expand for small m:

$$\frac{2e^{-\beta D}\left[\beta J z m + \frac{1}{3!}(\beta J z m)^3\right]}{1 + 2e^{-\beta D}\left[1 + \frac{1}{2}(\beta J z m)^2 + \frac{1}{4!}(\beta J z m)^4\right]} =$$

$$= \frac{2e^{-\beta D}}{1+2e^{-\beta D}} \frac{\beta J z m + \frac{1}{3!}(\beta J z m)^3}{1 + \frac{2e^{-\beta D}}{1+2e^{-\beta D}}\left(\frac{1}{2}(\beta J z m)^2 + \frac{1}{4!}(\beta J z m)^4\right)}$$

$$\simeq \frac{2e^{-\beta D}}{1+2e^{-\beta D}} \left\{\left(\beta J z m + \frac{1}{3!}(\beta J z m)^3\right)\cdot\left(1 - \frac{2e^{-\beta D}}{1+2e^{-\beta D}}\frac{1}{2}(\beta J z m)^2\right)\right\}$$

$$= \frac{2e^{-\beta D}}{1+2e^{-\beta D}} \left\{\beta J z m + \left[\frac{1}{6} - \frac{e^{-\beta D}}{1+2e^{-\beta D}}\right](\beta J z m)^3\right\} =$$

$$= \frac{2e^{-\beta D}}{1+2e^{-\beta D}} \left\{\beta J z m + \left(\frac{1-4e^{-\beta D}}{1+2e^{-\beta D}}\right)(\beta J z m)^3\right\}$$

We know that the slope at the origin must be >1 to have a phase transition

The slope is

$$\beta J z \frac{2e^{-\beta D}}{1+2e^{-\beta D}} > 1$$

$$T < \frac{Jz}{k_B} \frac{2e^{-\beta D}}{1+2e^{-\beta D}} \leq \frac{Jz}{k_B}$$

The critical temperature has decreased: the extra state ($s=0$) gives a bit more entropy that can disorder the system a bit

If $D \longrightarrow -\infty$ the $s=0$ becomes infinitely disfavoured $\Rightarrow$ Ising!

Is this the end of the story?

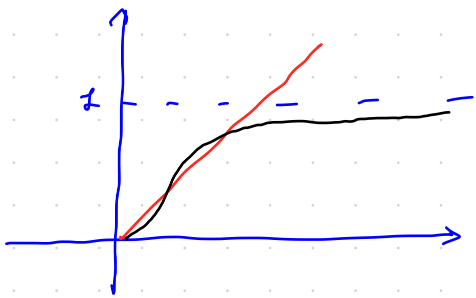
No!

If $\quad T > T_c = \dfrac{Jz}{k_B} \dfrac{2e^{-\beta D}}{1 + 2e^{-\beta D}}$

but if $\quad 4e^{-\beta D} < 1 \quad$ then

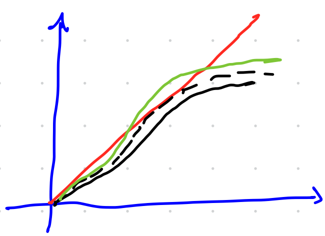
the cubic term has positive coefficient

This means, graphically

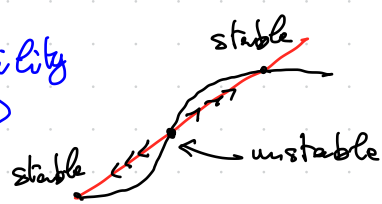


two extra solutions!

But they appear "suddenly" as D decreases



stability $\Longrightarrow$



First order phase transition!